QUESTION 1: EQUILIBRIUM POINTS OF A TIME-VARYING SYSTEM

$$\dot{x} = -x + t, \quad x(0) = x_0$$

(a) INSTANTANEOUS EQUILIBRIUM (TIME FROZEN).

FREEZE t AT A CONSTANT VALUE, THEN AN EQUILIBRIUM $\bar{x}$ SATISFIES

$$0 = -\bar{x} + t \quad \Rightarrow \quad \bar{x} = t$$

SO THE "INSTANTANEOUS" EQUILIBRIUM CURVE IS $\bar{x}(t) = t$

(b) SOLUTION OF THE ODE.

REWRITING AS $\dot{x} + x = t$ AND USING THE INTEGRATING FACTOR e^t:

$$\frac{d}{dt}(e^t x) = te^t \Rightarrow e^t x = (t-1)e^t + c \Rightarrow x(t) = t - 1 + ce^{-t}$$

SO $x(0) = x_0$ IMPLIES $x(t) = t - 1 + (x_0 + 1)e^{-t}$

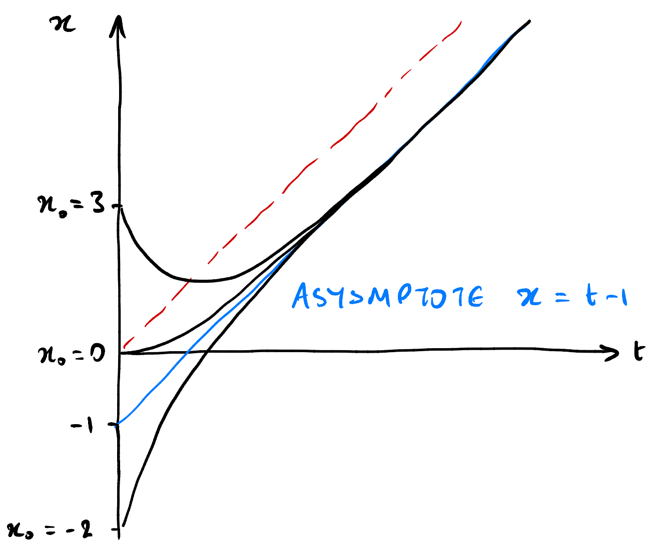
(c) EQUILIBRIUM OF THE NON AUTONOMOUS SYSTEM & TRAJECTORIES.

A TRUE (TIME-INDEPENDENT) EQUILIBRIUM MUST BE A CONSTANT $\bar{x}$ SATISFYING $0 = -\bar{x} + t$ FOR ALL t, WHICH IS IMPOSSIBLE BECAUSE t VARIES. THEREFORE THIS SYSTEM HAS NO EQUILIBRIUM.

FROM (b) EVERY TRAJECTORY SATISFIES

$$x(t) - (t-1) = e^{-t}(x_0 + 1) \xrightarrow[t \to \infty]{} 0.$$

THEREFORE EVERY TRAJECTORY IS ASYMPTOTIC TO THE LINE $x = t - 1$ (AND NOT TO THE INSTANTANEOUS CURVE $x = t$), ILLUSTRATING THAT "FREEZING TIME" CAN BE MISLEADING.



QUESTION 2: NATURE OF EQUILIBRIA OF MAPS

THIS QUESTION ASKS FOR THE STABLE, UNSTABLE, AND CENTRE SUBSPACES OF THE LINEAR MAP:

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad A = \begin{bmatrix} \lambda & 1 \\ 0 & \mu \end{bmatrix}, \quad \lambda \neq 0, \ \mu \neq 0$$

FIRST NOTE THAT $x_k = A^k x_0$, SO IF A IS DIAGONALISABLE: $A = V \Lambda V^{-1}$, Λ = DIAGONAL, THEN STABLE AND UNSTABLE INVARIANT DIRECTIONS CAN BE DETERMINED FROM Λ, V.

THUS THE UNSTABLE, STABLE, AND CENTRE SUBSPACES ARE SPANNED BY THE EIGENVECTORS WITH EIGENVALUES OF MODULUS <1, >1, AND $=1$, RESPECTIVELY.

THE EIGENVALUES OF A ARE λ AND μ (SINCE A IS UPPER TRIANGULAR).

FOR $\lambda \neq \mu$, WE CAN DEFINE THE EIGENVECTORS AS

$$v_\lambda = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_\mu = \begin{bmatrix} 1 \\ \mu - \lambda \end{bmatrix},$$

(a) $|\lambda|, |\mu| > 1$: $E^u = \mathbb{R}^2$, $E^s = \phi$, $E^c = \phi$

(b) $|\lambda|, |\mu| < 1$: $E^s = \mathbb{R}^2$, $E^u = \phi$, $E^c = \phi$

(c) $|\lambda| > 1$, $|\mu| < 1$:

$$E^u = \text{span}\{v_\lambda\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad E^s = \text{span}\{v_\mu\} = \text{span}\left\{\begin{bmatrix} 1 \\ \mu - \lambda \end{bmatrix}\right\},$$

$$E^c = \phi.$$

(d) $|\lambda| = 1$, $|\mu| > 1$:

$$E^c = \text{span}\{v_\lambda\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}, \quad E^u = \text{span}\{v_\mu\} = \text{span}\left\{\begin{bmatrix} 1 \\ \mu - \lambda \end{bmatrix}\right\},$$

$$E^s = \phi.$$

FOR THE SPECIAL CASE: $\lambda = \mu$ IN (a) OR (b): $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ IS DEFECTIVE (BECAUSE λ IS A REPEATED EIGENVALUE WITH ONLY 1 EIGENVECTOR) BUT THE CLASSIFICATION OF SUBSPACES IS UNCHANGED: $E^c = \phi$ AND

$$E^u = \mathbb{R}^2, \ E^s = \phi \ \text{ IF } |\mu| > 1 \quad \& \quad E^s = \mathbb{R}^2, \ E^u = \phi \ \text{ IF } |\mu| < 1$$

QUESTION 3 : LINEAR FLOWS

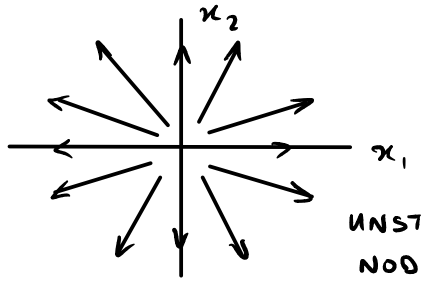
(a) WE ARE ASKED TO SOLVE $\dot{x} = Ax$, THEN LIST E^S, E^U, E^C (THE STABLE, UNSTABLE, AND CENTRE SUBSPACES DETERMINED BY $Re\,\lambda < 0, > 0, = 0$) AND DESCRIBE THE PHASE PORTRAIT.

(i) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} x(0)$

SUBSPACES:

$eig(A) = \{1, 1\} \Rightarrow Re\,\lambda_1, Re\,\lambda_2 > 0$

$E^u = \mathbb{R}^2, \quad E^s = \phi, \quad E^c = \phi$



UNSTABLE NODE

(ii) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \Rightarrow x(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} x(0)$

[OR SOLVE USING $\dot{x}_2 = x_2 \Rightarrow x_2(t) = e^t x_2(0)$

$\dot{x}_1 = x_1 + x_2 \Rightarrow \dot{x}_1 = x_1 + e^t x_2(0)$ (I.F. : e^{-t})

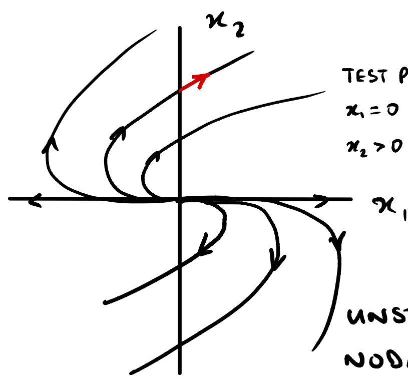
$\Rightarrow x_1(t) = e^t\left(x_1(0) + t\,x_2(0)\right)$]

SUBSPACES :

$eig(A) = \{1, 1\} \Rightarrow Re\,\lambda_1, Re\,\lambda_2 > 0$

SINGLE EIGENVECTOR $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$E^u = \mathbb{R}^2, \quad E^s = \phi, \quad E^c = \phi$



TEST POINT: $\left.\begin{matrix} x_1 = 0 \\ x_2 > 0 \end{matrix}\right\} \Rightarrow \begin{matrix} \dot{x}_1 > 0 \\ \dot{x}_2 > 0 \end{matrix}$

UNSTABLE NODE

NOTE: $E^u = \mathbb{R}^2$ EVEN THOUGH THERE IS ONLY ONE EIGENVECTOR

(iii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ EIGENVALUES : $(\lambda - 1)(\lambda - 2)(-1-\lambda) = 0 \Rightarrow \lambda = \{1, 2, -1\}$

$\therefore \dot{x} = Ax \Rightarrow x(t) = c_1 e^t v_1 + c_2 e^{2t} v_2 + c_3 e^{-t} v_{-1}$

FOR CONSTANTS c_1, c_2, c_3

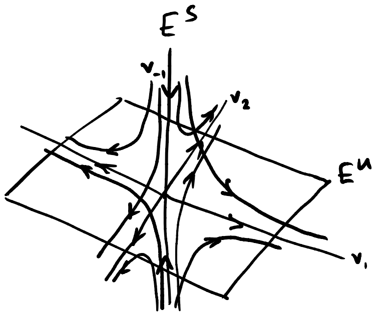
EIGENVECTORS: $v_1 = \begin{bmatrix} 1 \\ -1 \\ \frac{1}{2} \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_{-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

SUBSPACES:

$$E^u = \text{span}\{v_1, v_2\}, \quad E^s = \text{span}\{v_{-1}\}, \quad E^c = \phi$$

PHASE PORTRAIT:

3D SADDLE WITH TRAJECTORIES ATTRACTED TO 0 ALONG v_{-1} AXIS (E^s) AND REPELLED WITHIN THE 2D PLANE SPANNED BY v_1, v_2 (E^u)



SOLUTION:

$$x(t) = e^{At}x(0) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} e^t & & \\ & e^{2t} & \\ & & e^{-t} \end{bmatrix}}_{e^{\Lambda t}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}}_{V^{-1}} x(0)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} & e^{2t} & 0 \\ -\frac{1}{2}e^{-t} & 0 & e^{-t} \end{bmatrix} x(0) = \begin{bmatrix} e^t & 0 & 0 \\ e^{2t} - e^t & e^{2t} & 0 \\ \frac{1}{2}(e^t - e^{-t}) & 0 & e^{-2t} \end{bmatrix} x(0)$$

(iv) $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ EIGENVALUES: $(-1-\lambda)^2 - 1 = 0 \Rightarrow \lambda = \{0, -2\}$

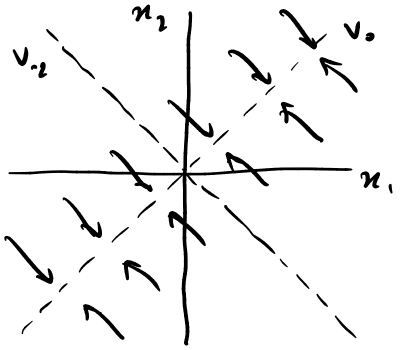
EIGENVECTORS: $v_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_{-2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

SOLUTION: $x(t) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} x(0) = \frac{1}{2} \begin{bmatrix} 1 + e^{-2t} & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{bmatrix} x(0)$

SUBSPACES:

$$E^s = \text{span}\{v_{-2}\}, \quad E^c = \text{span}\{v_0\}$$

$$E^u = \phi$$



(b) LET $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. THE QUESTION ASKS US TO PROVE $e^{A\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

METHOD 1 (USING DIAGONALISATION): $\lambda = \pm i$, $A = \begin{bmatrix} -1 & 1 \\ -i & -i \end{bmatrix} \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix}$

$$\therefore\ e^{A\theta} = \begin{bmatrix} -1 & 1 \\ -i & -i \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}\frac{1}{2}\begin{bmatrix} -1 & i \\ 1 & i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -i & -i \end{bmatrix}\begin{bmatrix} -e^{-i\theta} & ie^{-i\theta} \\ e^{i\theta} & ie^{i\theta} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e^{-i\theta}+e^{i\theta} & i(e^{i\theta}+e^{-i\theta}) \\ i(e^{-i\theta}-e^{i\theta}) & e^{-i\theta}+e^{i\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

METHOD 2 (USING SERIES EXPANSION) : $A^2 = -I,\ A^3 = -A,\ A^4 = I$ ETC.

$$\therefore\ e^{A\theta} = \sum_{n=0}^{\infty}\frac{\theta^n}{n!}A^n = \sum_{k=0}^{\infty}\frac{(\theta A)^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(\theta A)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}\cdot I + \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k+1}}{(2k+1)!}\cdot A = \cos\theta\, I + \sin\theta\, A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(c) ASSUME $\{v_1, \dots, v_n\}$ ARE LINEARLY INDEPENDENT, THEN $A = V\Lambda V^{-1}$, WHERE $\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$, $V = [v_1 \cdots v_n]$, AND HENCE $\dot{x} = Ax$ HAS SOLUTION

$$x(t) = e^{At}x(0) = Ve^{\Lambda t}V^{-1}x(0)$$

BUT $x(0) = V\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, SO $x(t) = Ve^{\Lambda t}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1 e^{\lambda_1 t}v_1 + \dots + c_n e^{\lambda_n t}v_n$

(d) LET $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, THEN THE CHARACTERISTIC POLYNOMIAL IS

$$\det(\lambda I - A) = 0 \iff (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

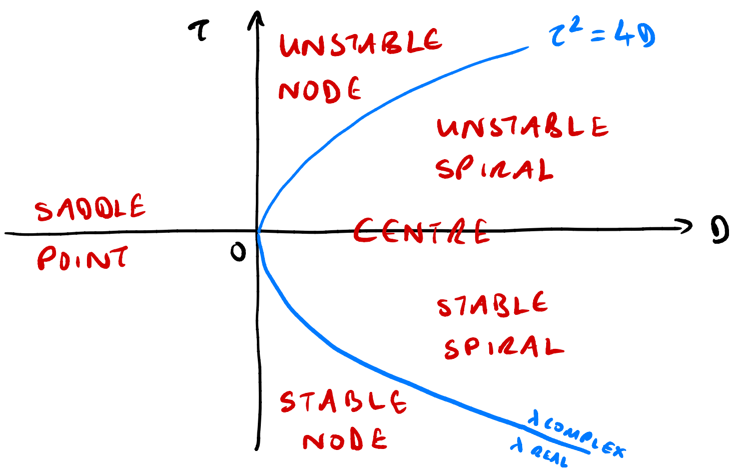
$$\iff \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$

$$\iff \lambda^2 - \underbrace{\mathrm{tr}(A)}_{=\tau}\lambda + \underbrace{\det(A)}_{=D} = 0$$

SO $\lambda^2 - \tau\lambda + D = 0$

$$\Rightarrow \lambda = \frac{\tau}{2} \pm \frac{\sqrt{\tau^2 - 4D}}{2}$$

$$\Rightarrow \begin{cases} \mathrm{Im}(\lambda) = 0 & \text{if } \tau^2 \geq 4D \\ \mathrm{Re}(\lambda) \leq 0 & \text{if } \tau \leq 0 \end{cases}$$



QUESTION 4: NONLINEAR SYSTEMS

(a) $\dot{x}_1 = (3 - x_1 - x_2)x_1$, $\dot{x}_2 = (x_1 - 1)x_2$ } EQUILIBRIUM POINTS: $\dot{x}_1 = 0 \Rightarrow x_1 = 0$ OR $x_1 + x_2 = 3$

$\dot{x}_2 = 0 \Rightarrow x_1 = 1$ OR $x_2 = 0$

$\therefore (x_1, x_2) = (0,0), (1,2), (3,0)$

LINEARISATION: $J(x_1, x_2) = \begin{bmatrix} 3 - 2x_1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$

AT $(0,0)$: $J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda = 3, -1$, $v_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_{-1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

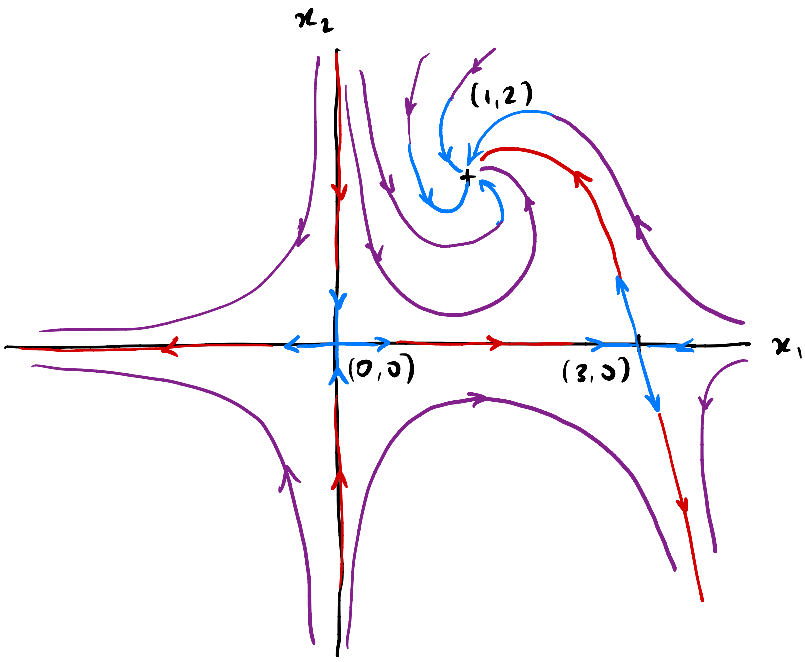
SADDLE POINT

AT $(1,2)$: $J(1,2) = \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} \Rightarrow (\lambda + 1)\lambda + 2 = 0$, $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$

STABLE SPIRAL (ANTI-CLOCKWISE)

AT $(3,0)$: $J(3,0) = \begin{bmatrix} -3 & -3 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda = 2, -3$, $v_2 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, $v_{-3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

SADDLE POINT



TO SKETCH:

① DRAW THE LOCAL BEHAVIOUR AROUND EQUILIBRIA (BLUE)

② DRAW THE SEPARATRICES (RED)

③ ADD MORE TRAJECTORIES (PURPLE)

(SEE ALSO THE MATHEMATICA NOTEBOOK)

(b) EQUILIBRIUM POINTS :

$$\left.\begin{aligned}\dot{x}_1 &= x_1(x_1+x_2) = 0\\ \dot{x}_2 &= x_1\left(x_1+\tfrac{1}{2}x_2\right) = 0\end{aligned}\right\} \Rightarrow \quad (x_1, x_2) = (0, 0) \quad \text{(ONLY)}$$

LINEARISATION : $J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ NON-HYPERBOLIC EQUILIBRIUM

INVARIANTS :

$$\left.\begin{aligned} x_1 = 0 &\Rightarrow \dot{x}_1 = 0 \\ x_2 = 0 &\Rightarrow \dot{x}_2 = 0\end{aligned}\right\} \Rightarrow$$

x_1-AXIS AND x_2-AXIS ARE INVARIANT (THEY CONTAIN TRAJECTORIES)

ON $x_1 = 0$: $\dot{x}_1 = 0$, $\dot{x}_2 = \frac{1}{2}x_2^2 \geq 0 \Rightarrow$ SOLUTIONS WITH $x_2 > 0$ GO TO $+\infty$

" $x_2 < 0$ GO TO 0^-

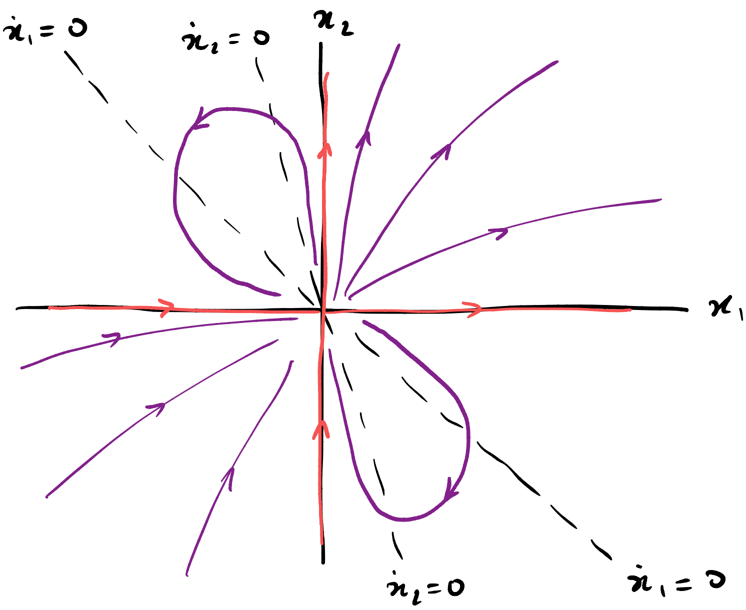
ON $x_2 = 0$: $\dot{x}_2 = 0$, $\dot{x}_1 = x_1^2 \geq 0 \Rightarrow$ SOLUTIONS WITH $x_1 > 0$ GO TO $+\infty$

" $x_1 < 0$ GO TO 0^-

$x_1 + x_2 = 0 \Rightarrow \dot{x}_1 = 0$

$x_1 + \frac{1}{2}x_2 = 0 \Rightarrow \dot{x}_2 = 0$

PHASE-PLANE SKETCH :



(SEE ALSO THE MATHEMATICA NOTEBOOK)

(c) $\dot{x}_1 = x_1^2$ EQUILIBRIUM: $\dot{x}_1 = 0 \Rightarrow x_1 = 0$,

$\dot{x}_2 = x_2$ $\dot{x}_2 = 0 \Rightarrow x_2 = 0$ $\therefore (x_1, x_2) = (0,0)$

LINEARISATION: $J(x_1,x_2) = \begin{bmatrix} 2x_1 & 0 \\ 0 & 1 \end{bmatrix}$, $J(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

EIGENVALUES: $\lambda = \{0, 1\}$ $v_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\therefore E^u = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ $E^c = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ $\Rightarrow$ NON-HYPERBOLIC EQUILIBRIUM

INVARIANTS: $x_1 = 0 \Rightarrow \dot{x}_1 = 0$

$x_2 = 0 \Rightarrow \dot{x}_2 = 0$

SINCE THIS SYSTEM IS DECOUPLED, WE CAN SOLVE FOR $x_1(t)$, $x_2(t)$ DIRECTLY:

$\dot{x}_1 = x_1^2$: SOLVE BY SEPARATING VARS.: $\int \frac{dx_1}{x_1^2} = t + c_1 \Rightarrow x_1(t) = \frac{-1}{c_1 + t}$

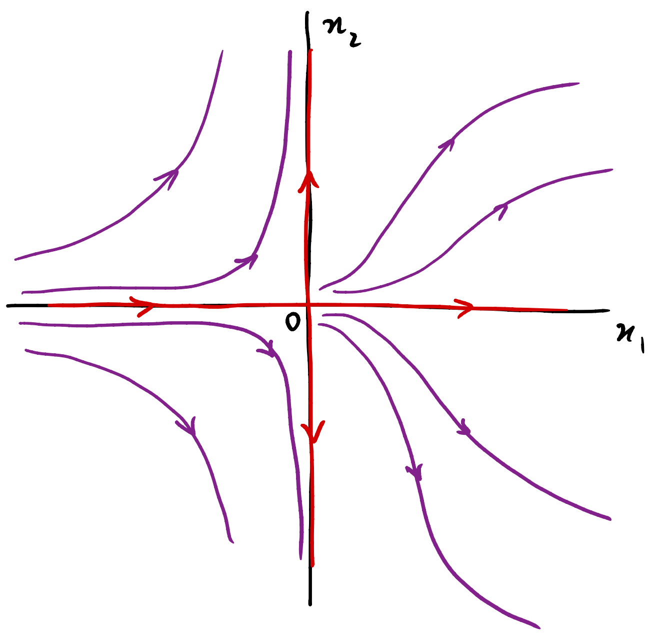
$\dot{x}_2 = x_2$: SOLVE AS LINEAR ODE: $x_2(t) = c_2 e^t$

INITIAL CONDITION $(x_1(0), x_2(0)) \Rightarrow c_1 = -1/x_1(0)$, $c_2 = x_2(0)$

SOLUTION: $x_1(t) = \frac{1}{1/x_1(0) - t}$ $\therefore x_1 \to \infty$ AS $t \to 1/x_1(0)$ IF $x_1(0) > 0$

$x_1 \to 0^-$ AS $t \to \infty$ IF $x_1(0) < 0$

$x_2(t) = x_2(0)\, e^t$ $\therefore x_2 \to \pm\infty$ AS $t \to \infty$ IF $x_2(0) \gtrless 0$



(SEE ALSO THE MATHEMATICA NOTEBOOK)

(d) $\left.\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1^2\end{aligned}\right\} \Rightarrow \dfrac{dx_1}{dx_2} = \dfrac{x_2}{x_1^2} \Rightarrow \int x_1^2\, dx_1 = \int x_2\, dx_2$

$\therefore$ ALL TRAJECTORIES LIE ON THE CURVES $\frac{1}{3}x_1^3 - \frac{1}{2}x_2^2 = \text{CONSTANT}$

QUALITATIVE BEHAVIOUR

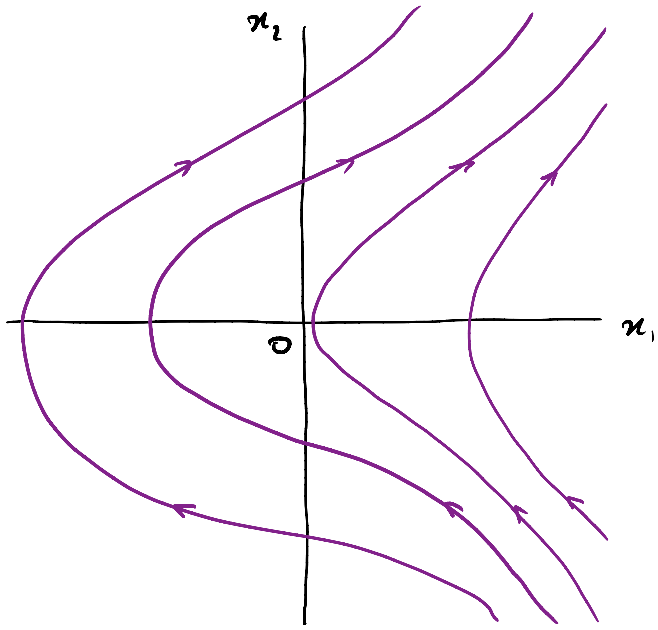
EQUILIBRIUM: $(x_1, x_2) = (0,0)$

LINEARISATION: $J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

EIGENVALUES: $\lambda = 0$ (REPEATED) $\therefore E^c = \mathbb{R}^2$, NON-HYPERBOLIC EQUILIBRIUM

NULLCLINES (WHERE $\dot{x}_1 = 0$ OR $\dot{x}_2 = 0$): $x_1 = 0 \Rightarrow \dot{x}_2 = 0$

$x_2 = 0 \Rightarrow \dot{x}_1 = 0$



ALSO, FOR ALL $x_1 \neq 0$, $\dot{x}_2 > 0$

(SEE ALSO THE MATHEMATICA NOTEBOOK)

QUESTION 5: POLAR TRANSFORMATION

(a) $r^2 = x_1^2 + x_2^2$, $\tan\theta = x_2/x_1$

$\therefore 2r\dot{r} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$

$\Rightarrow \dot{r} = \dfrac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}$

$\sec^2\theta \cdot \dot{\theta} = \dfrac{\dot{x}_2}{x_1} - \dfrac{x_2\dot{x}_1}{x_1^2} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}$

$\dot{\theta} = \dfrac{x_1\dot{x}_2 - x_2\dot{x}_1}{r^2}$

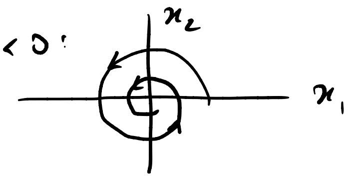
(SINCE $\sec^2\theta = 1 + \tan^2\theta = r^2/x_1^2$)

(1b)
$$\dot{x}_1 = -x_2 + a x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + a x_2 (x_1^2 + x_2^2)$$
RE-WRITE IN POLAR CO-ORD.S

$$\dot{r} = \frac{-x_1 x_2 + a x_1^2 r^2 + x_1 x_2 + a x_2^2 r^2}{r^2} = a r^3$$

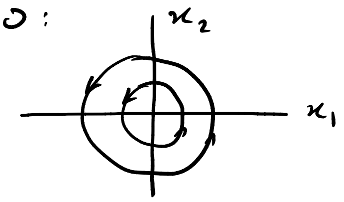
$$\dot{\theta} = \frac{x_1^2 + a x_1 x_2 r^2 + x_2^2 - a x_1 x_2 r^2}{r^2} = 1$$

$a < 0$:

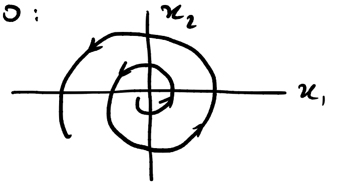


$\Rightarrow$ STABLE SPIRAL IF $a < 0$
CENTRE IF $a = 0$
UNSTABLE SPIRAL IF $a > 0$

$a = 0$:



$a > 0$:



QUESTION 6 : NONLINEAR CENTRES

(a) LET $x_1 = x$, THEN $\ddot{x} = f(x)$ IS EQUIVALENT TO

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(x_1)$$

(b) LET $\frac{dV}{dt} = -f(x_1)\dot{x}_1 + x_2 \dot{x}_2$ WHERE $\dot{x}_1 = x_2$, $\dot{x}_2 = f(x_1)$,

THEN $\frac{dV}{dt} = -f(x_1) x_2 + x_2 f(x_1) = 0$

AND $V = -\int f(x_1)\, dx_1 + \frac{1}{2} x_2^2$ IS A NON-CONSTANT FUNCTION

HENCE V IS CONSERVED ($\dot{V} = 0$) ALONG THE SYSTEM TRAJECTORIES

(c)
$$\left.\begin{aligned} \dot{z}_1 &= -z_2 - z_2^3 \\ \dot{z}_2 &= z_1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_1^3 \end{aligned}\right.$$
WITH $x_1 = z_2$, $x_2 = z_1$

$$\therefore \quad V(z_1, z_2) = \frac{1}{2} z_1^2 + \int^{z_2} (z + z^3)\, dz = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{4} z_2^4 .$$

THIS V IS POSITIVE DEFINITE. SO, FOR SUFFICIENTLY SMALL c, THE CURVE $\{(z_1, z_2) : V = c\}$ CONTAINS SYSTEM TRAJECTORIES AND IS A CLOSED CURVE AROUND (0,0)

$\Rightarrow$ THE ORIGIN IS A NONLINEAR CENTRE

QUESTION 7 : LYAPUNOV FUNCTIONS

(a)
$$\dot{x} = -y - x^2(x^2+y^2)$$
$$\dot{y} = x - y^2(x^2+y^2)$$

USE $V(x,y) = \frac{1}{2}(x^2+y^2)$:

$$\dot{V}(x,y) = x\dot{x} + y\dot{y}$$
$$= -xy - x^3(x^2+y^2) + xy - xy^2(x^2+y^2)$$
$$= -(x^2+y^2)$$

WE HAVE $\dot{V} < 0 \quad \forall (x,y) \neq (0,0)$

& $V > 0 \quad \forall (x,y) \neq (0,0)$

& $V(0,0) = \dot{V}(0,0) = 0$

& $V(x,y) \to \infty$ AS $x^2+y^2 \to \infty$

$\Rightarrow$ $(x,y) = (0,0)$ IS <u>GLOBALLY ASYMPTOTICALLY STABLE</u>

(b). SYSTEM : $\dot{x} = y$

$\dot{y} = x - x^3 - \gamma y, \quad \gamma > 0$ (DAMPED DUFFING OSCILLATOR)

LET $V(x,y) = 2y^2 - 2x^2 + x^4$

TO MAKE $V(\pm 1, 0) = 0$, ADD A CONSTANT TERM:

$$V'(x,y) = 2y^2 + x^4 - 2x^2 + 1 = 2y^2 + (x^2-1)^2$$

THEN

$$\dot{V}'(x,y) = 4y\dot{y} + 2x.2(x^2-1).\dot{x}$$
$$= 4yx(1-x^2) - 4\gamma y^2 + 4xy(x^2-1)$$
$$= -4\gamma y^2$$

WE HAVE: $\dot{V}'(x,y) \leq 0 \quad \forall (x,y)$

$V'(x,y) > 0 \quad \forall (x,y) \neq (\pm 1, 0)$

$V'(\pm 1, 0) = 0$

$\Rightarrow$ $(x,y) = (\pm 1, 0)$ IS <u>STABLE</u>

NOTE THAT LYAPUNOV'S METHOD APPLIED TO $V'(x,y)$ DOES NOT SHOW ASYMPTOTIC STABILITY OF $(x,y) = (\pm 1, 0)$

BECAUSE WE HAVE NOT SHOWN THAT $\dot{V}'(x,y) < 0 \;\forall\; (x,y) \neq (\pm 1, 0)$

$\hookrightarrow$ E.G. $\dot{V}'(x,0) = 0 \;\forall\; x$

BUT WE CAN PROVE THAT $(\pm 1, 0)$ IS GLOBALLY ASYMPTOTICALLY STABLE BY APPLYING LASALLE'S INVARIANCE PRINCIPLE USING $V'(x,y)$ — SEE LECTURE 5

(c). SYSTEM; $\dot{x}_1 = -x_1 + 2x_2^3 - 2x_2^4$

$$\dot{x}_2 = -x_1 - x_2 + x_1 x_2$$

LET $V(x_1, x_2) = x_1^{\alpha_1} + k x_2^{\alpha_2}$ WITH $\alpha_1, \alpha_2 \geq 2$, $k > 0$ SO THAT $V \geq 0$

THEN
$$\begin{aligned}
\dot{V}(x_1, x_2) &= \alpha_1 x_1^{\alpha_1 - 1} \dot{x}_1 + k \alpha_2 x_2^{\alpha_2 - 1} \dot{x}_2 \\
&= \alpha_1 x_1^{\alpha_1 - 1}(-x_1 + 2x_2^3 - 2x_2^4) + k\alpha_2 x_2^{\alpha_2 - 1}(-x_1 - x_2 + x_1 x_2) \\
&= -\alpha_1 x_1^{\alpha_1} - k\alpha_2 x_2^{\alpha_2} + (2\alpha_1 x_1^{\alpha_1 - 2} - k\alpha_2 x_2^{\alpha_2 - 4}) \cdot x_1 x_2^3 \\
&\quad + (k\alpha_2 x_2^{\alpha_2 - 4} - 2\alpha_1 x_1^{\alpha_1 - 2}) \cdot x_1 x_2^4
\end{aligned}$$

3RD & 4TH TERMS ARE SIGN-INDEFINITE, SO SET THEM TO 0 BY DEFINING:

$\alpha_1 = 2$, $\alpha_2 = 4$ $\Rightarrow$ TERMS IN ( ) ARE INDEPENDENT OF x_1, x_2

$k = 1$ $\Rightarrow$ TERMS IN ( ) ARE ZERO

$\therefore\; V(x_1, x_2) = x_1^2 + x_2^4 \Rightarrow \dot{V}(x_1, x_2) = -2x_1^2 - 4x_2^4$

$$\left.\begin{array}{ll}
\text{WE HAVE} & \dot{V} < 0 \quad \forall\, (x,y) \neq (0,0) \\
\& & V > 0 \quad \forall\, (x,y) \neq (0,0) \\
\& & V(0,0) = \dot{V}(0,0) = 0 \\
\& & V(x,y) \to \infty \text{ AS } x^2 + y^2 \to \infty
\end{array}\right\} \Rightarrow$$

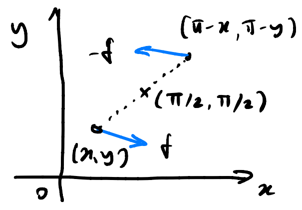
$(x,y) = (0,0)$ IS GLOBALLY ASYMPTOTICALLY STABLE

QUESTION 8: HAMILTONIAN AND GRADIENT SYSTEMS

(a) $\left.\begin{aligned} \dot{x} &= 2\cos x + \cos y \\ \dot{y} &= \cos x + 2\cos y \end{aligned}\right\} \Rightarrow (\dot{x}, \dot{y}) = f(x,y)$

LET $(u,v) = (\pi - x, \pi - y)$, THEN $(\dot{u}, \dot{v}) = -f(x,y)$

SINCE $\cos(\pi - u) = -\cos u$, $\cos(\pi - v) = -\cos v$



THEREFORE THE PHASE PORTRAIT HAS CENTRAL SYMMETRY ABOUT THE EQUILIBRIUM AT $(x_1, x_2) = (\pi/2, \pi/2)$

TO CHECK WHETHER IT IS A CONSERVATIVE SYSTEM, CONSIDER

(i). $\nabla . f(x,y) = \frac{\partial}{\partial x}(2\cos x + \cos y) + \frac{\partial}{\partial y}(\cos x + 2\cos y) = -2\sin x - 2\sin y$

(ii). THE LINEARISATION ABOUT $(\pi/2, \pi/2)$ HAS EIGENVALUES $\{-3, -1\}$

(i) $\Rightarrow$ $\nabla . f \neq 0$ IF $(x,y) \neq (j\pi, k\pi)$ FOR INTEGER j, k $\Rightarrow$ NOT A HAMILTONIAN SYSTEM

(BUT IT COULD BE CONSERVATIVE IF $\nabla . f = 0$ ON AVERAGE)

(ii) $\Rightarrow$ THIS IS AN ATTRACTING EQUILIBRIUM WITH NO PERIODIC ORBITS

HENCE THE SYSTEM CAN'T BE CONSERVATIVE

(b) SYSTEM: $\left.\begin{aligned} \dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= x_1 \cos x_2 \end{aligned}\right\}$ IS A GRADIENT SYSTEM BECAUSE

$$\frac{\partial}{\partial x_2}(\sin x_2) = \frac{\partial}{\partial x_1}(x_1 \cos x_2)$$

POTENTIAL: $\left.\begin{aligned} \frac{\partial V}{\partial x_1} &= \sin x_2 \\ \frac{\partial V}{\partial x_2} &= x_1 \cos x_2 \end{aligned}\right\} \Rightarrow V = x_1 \sin x_2$

RELATED HAMILTONIAN SYSTEM; LET $H = V = x_1 \sin x_2$ THEN

$$\dot{x}_1 = -\frac{\partial H}{\partial x_2} = -x_1 \cos x_2, \quad \dot{x}_2 = \frac{\partial H}{\partial x_1} = \sin x_2$$

Examples Sheet 1: Solutions for Question 4

In[ ]:=
```
flow4a[x1_, x2_] := {x1 * (3 - x1 - x2), x2 * (x1 - 1)}
```

In[ ]:=
```
Jaco[x1_, x2_] :=
{{D[flow4a[x, x2][[1]], x] /. x → x1, D[flow4a[x1, y][[1]], y] /. y → x2},
  {D[flow4a[x, x2][[2]], x] /. x → x1, D[flow4a[x1, y][[2]], y] /. y → x2}}
```

In[ ]:=
```
MatrixForm[Jaco[x1, x2]]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 3 - 2\,x1 - x2 & -x1 \\ x2 & -1 + x1 \end{pmatrix}$$

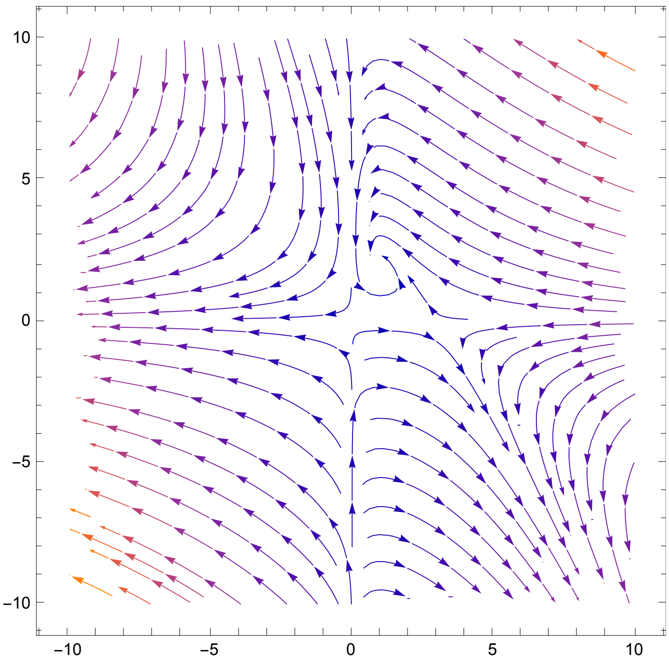
In[ ]:=
```
Eigenvectors[Jaco[3, 0]]
```

Out[ ]=

```
{{1, 0}, {-3, 5}}
```

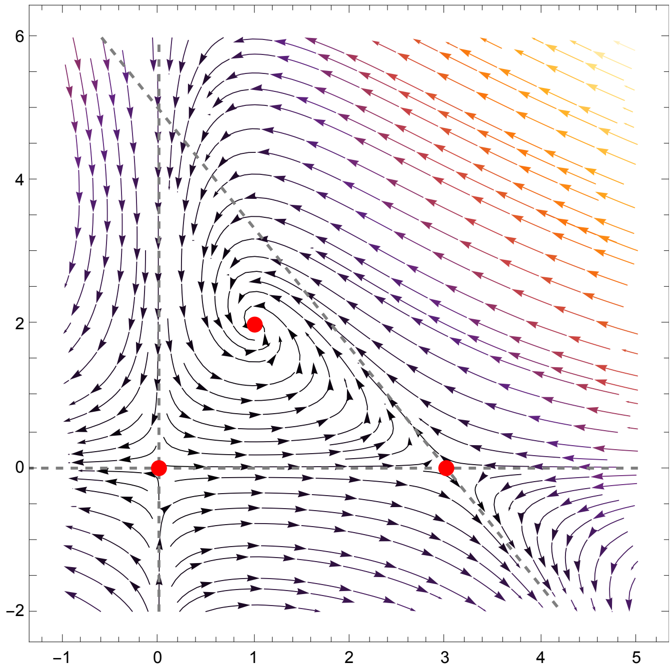
In[ ]:=
```
StreamPlot[flow4a[x1, x2], {x1, -10, 10}, {x2, -10, 10}]
```

Out[ ]=



In[ ]:=
```
Show[StreamPlot[flow4a[x1, x2], {x1, -1, 5}, {x2, -2, 6}, PlotTheme → "Detailed",
   StreamColorFunction → "SunsetColors"], Plot[{0, 10^6 * x, -5 / 3 * (x - 3)},
   {x, -2, 5}, PlotRange → {{-1, 5}, {-2, 6}}, Frame → True, AspectRatio → 8 / 6,
   PlotStyle → {{Gray, Dashed}, {Gray, Dashed}, {Gray, Dashed}}],
  ListPlot[{Style[{0, 0}, {Red, PointSize → 0.025}], Style[{3, 0},
     {Red, PointSize → 0.025}], Style[{1, 2}, {Red, PointSize → 0.025}]}]]
```

Out[ ]=



In[ ]:=
```
flow4b[x1_, x2_] := {x1^2 + x1 * x2, x2^2 / 2 + x1 * x2}
```

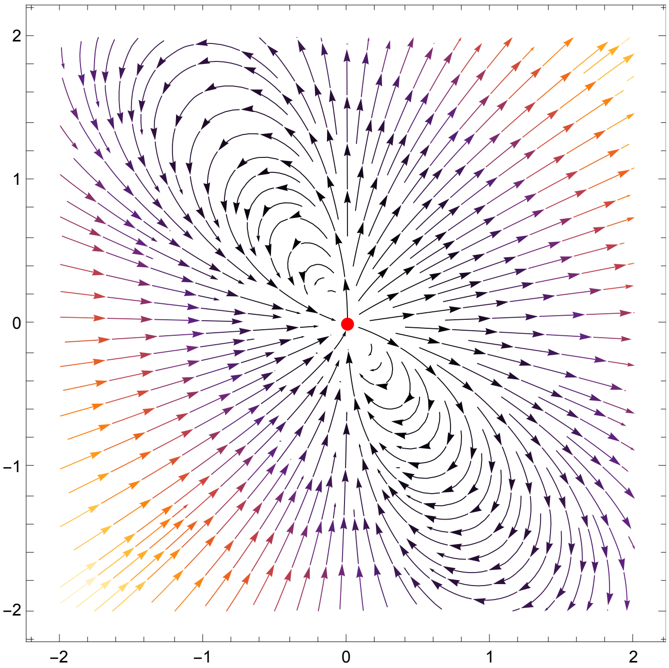
In[ ]:=
```
Solve[{flow4b[x1, x2][[1]] == 0, flow4b[x1, x2][[2]] == 0}, {x1, x2}]
```

Out[ ]=

```
{{x1 → 0, x2 → 0}, {x1 → 0, x2 → 0}}
```

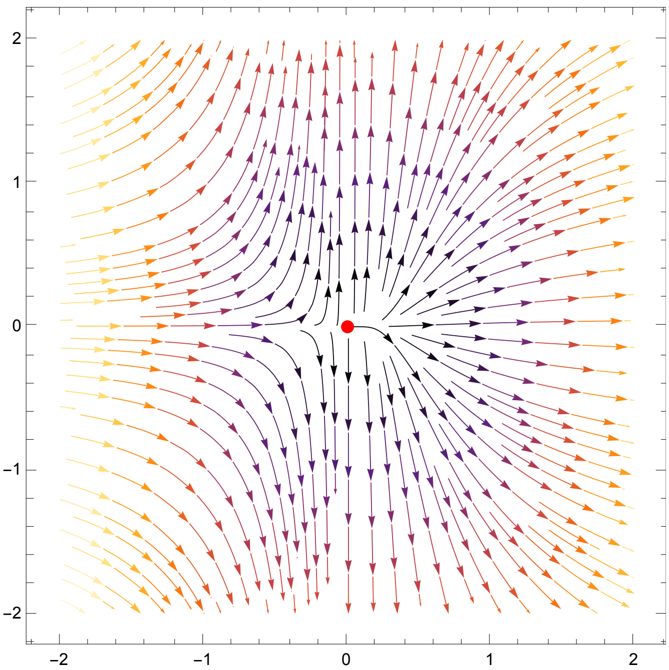
In[ ]:=
```
Show[StreamPlot[flow4b[x1, x2], {x1, -2, 2}, {x2, -2, 2},
   PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
 ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

Out[ ]=



In[ ]:=
```
flow4c[x1_, x2_] := {x1^2, x2}
Show[StreamPlot[flow4c[x1, x2], {x1, -2, 2}, {x2, -2, 2},
   PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
 ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

Out[ ]=



In[ ]:=
```
flow4d[x1_, x2_] := {x2, x1^2}
```

In[ ]:=
```
Show[StreamPlot[flow4d[x1, x2], {x1, -2, 2}, {x2, -2, 2},
   PlotTheme → "Detailed", StreamColorFunction → "SunsetColors"],
  ListPlot[{Style[{0, 0}, {Red, PointSize → 0.02}]}]]
```

Out[ ]=